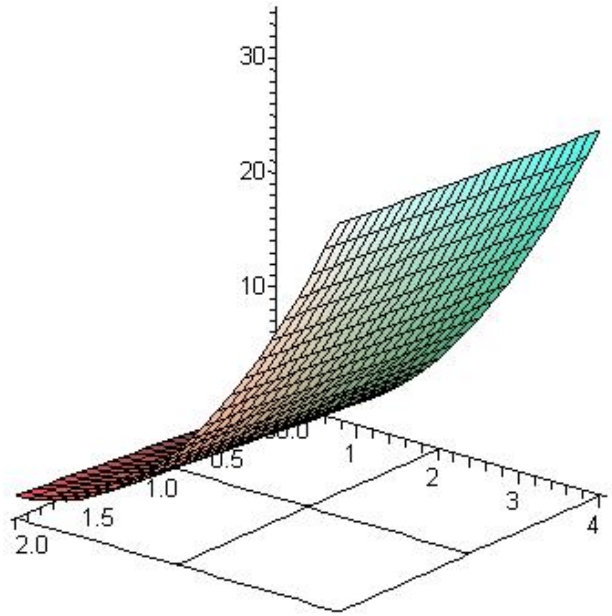
15.1-15.2 Intro to double Integrals

In all of ch. 15, you are given two things:

1. A surface: z = f(x,y)
2. A region drawn on the *xy*-plane.



Example: Let's ***estimate*** the volume

1. under $z = f(x,y) = x + 2y^2$
2. above R = [0,2] x [0,4]

   = {(x,y) : 0 ≤ x ≤ 2, 0 ≤ y ≤ 4 }

To estimate:

- Break R into *m* columns and *n* rows
- Over each sub-region, use function to get a rectangular box.
- Add up rectangular box volumes.

Notes and Observations

Formally,

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

= \`signed' volume between *f(x,y)* and *R*.

If *f(x,y)* is above the *xy*-plane it is *positive*.
If *f(x,y)* is below the *xy*-plane it is *negative*.

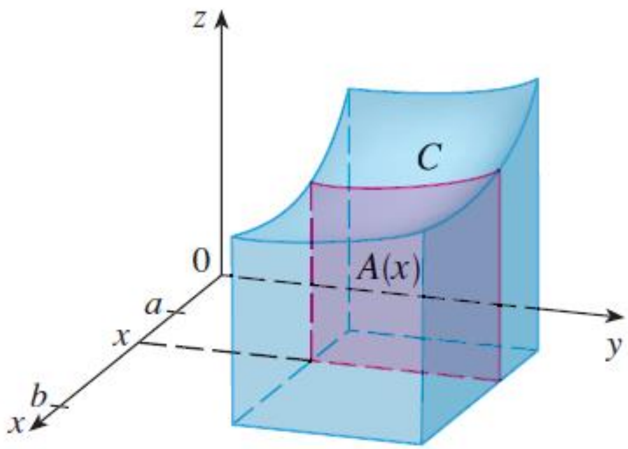
ΔA = area of base = $\Delta x \Delta y = \Delta y \Delta x$
$f(x_{ij}, y_{ij})\Delta A$ = (height)(area of base)
= volume of one approximating box

Units of $\iint_R f(x,y)dA$ are
(units of f(x,y))(units of x)(units of y)

Iterated Integrals

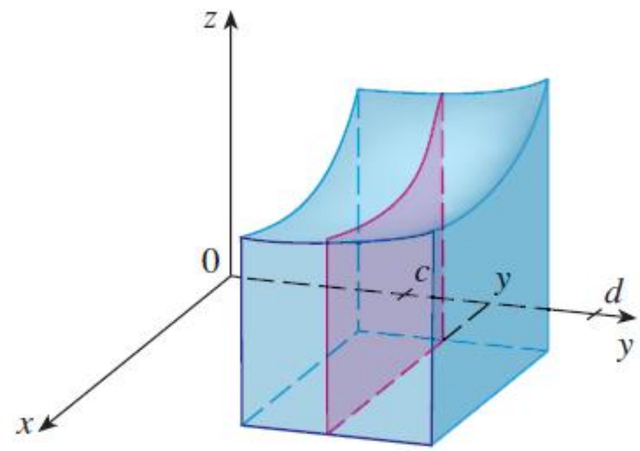
If you fix x: The area under this curve is

$$\int_c^d f(x,y)dy = \text{"cross sectional area under the surface at this fixed } x \text{ value"}$$



If you fix y: The area under this curve

$$\int_a^b f(x,y)dx = \text{"cross sectional area under the surface at this fixed } y \text{ value"}$$



From Math 125,

$$\text{Vol} = \int_a^b \text{Area(x)}dx = \int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

$$\text{Vol} = \int_c^d \text{Area}(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

Quick Example: Evaluate

$$(a) \int_{2}^{6} \left(\int_{1}^{8} y \, dx \right) dy$$

$$(b) \int_{2}^{6} \left(\int_{1}^{8} 1 \, dx \right) dy$$

Examples (like 15.2 HW):

1. Find the volume under
   $z = x + 2y^2$ and
   above $0 \leq x \leq 2, \quad 0 \leq y \leq 4$

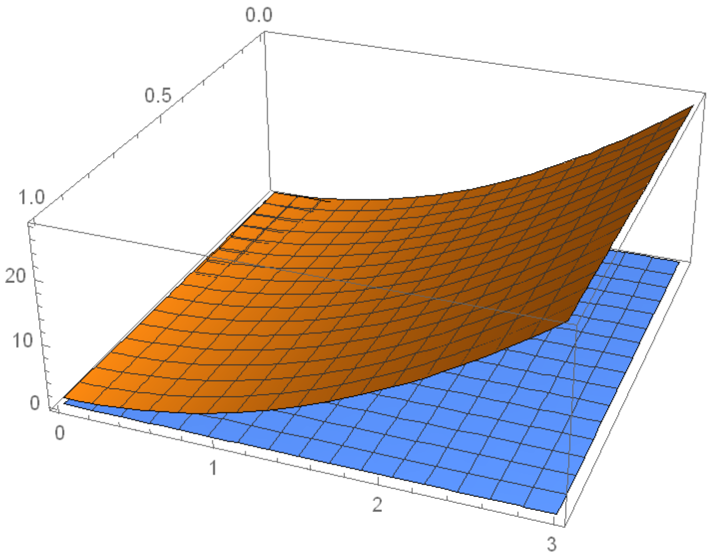
2. $\int_{0}^{3}\int_{0}^{1} 2xy\sqrt{x^2+y^2}\,dx\,dy$

15.2 Double Integrals over General Regions

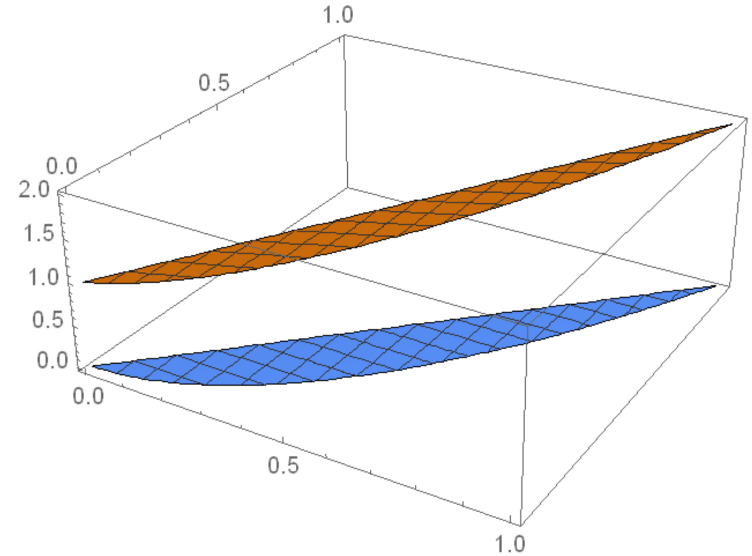
In 15.2, we discuss regions, *R*, other than rectangles.

Type 1 Regions (Top/Bot)	Type 2 Regions (Left/Right)
Given *x* in the range, $a \le x \le b$, we have $g_1(x) \le y \le g_2(x)$	Given *y* in the range, $c \le y \le d$, we have $h_1(y) \le x \le h_2(y)$
$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx$	$\int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy$

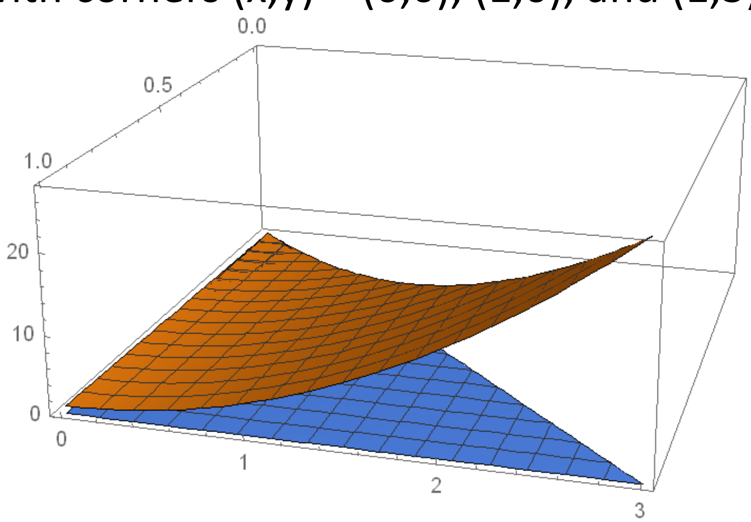
The surface $z = x + 3y^2$ over the rectangular region $R = [0,1] \times [0,3]$



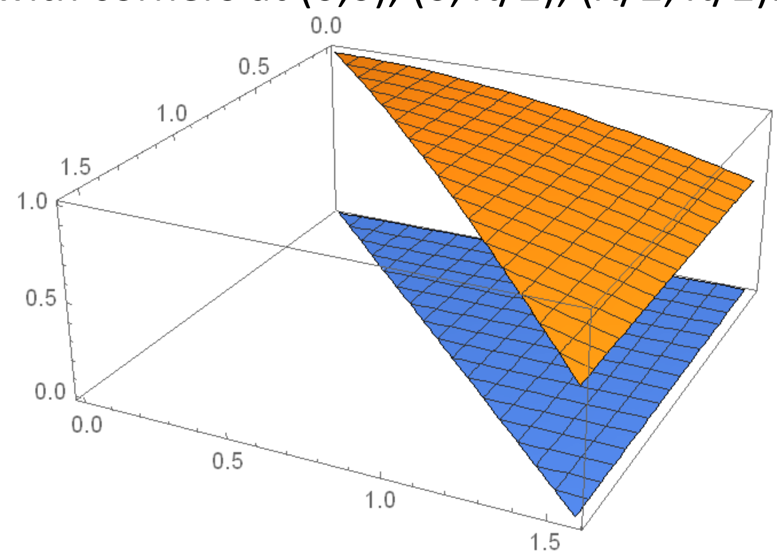
The surface $z = x + 1$ over the region bounded by $y = x$ and $y = x^2$.



The surface $z = x + 3y^2$ over the triangular region with corners $(x,y) = (0,0)$, $(1,0)$, and $(1,3)$.



The surface $z = \sin(y)/y$ over the triangular region with corners at $(0,0)$, $(0, \pi/2)$, $(\pi/2, \pi/2)$.



Examples:

1. Let D be the triangular region in the xy-plane with corners (0,0), (1,0), (1,3).

   Evaluate $\iint\limits_D x + 3y^2 \, dA$

2. Find the volume of the solid bounded by the surfaces $z = x + 1, y = x^2$, $y = 2x, z = 0$.

3. Draw the region of integration for

$$\int_0^{\pi/2}\int_x^{\pi/2} \frac{\sin(y)}{y}\, dy\, dx$$

then switch the order of integration.

4. Switch the order of integration for

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\, dy dx$$

Setting up a problem given in "words":

1. Find integrand

Solve for "z" anywhere you see it.
If there are two z's, then set up two double integrals (subtract at end).

2. Region?

Graph the region in the *xy*-plane.

a) Graph given *x* and *y* constraints.
b) And find the xy-curves where the surfaces (the z's) intersect.

Examples (directly from HW):

HW 15.2: Find the volume enclosed by $z = 4x^2 + 4y^2$ and the planes $x = 0$, $y = 2$, $y = x$, and $z = 0$.

HW 15.3:

Find the volume below $z = 18 - 2x^2 - 2y^2$

and above the xy-plane.

HW 15.3:

Find the volume enclosed by

$-x^2 - y^2 + z^2 = 22$ and $z = 5$.

HW 15.3:

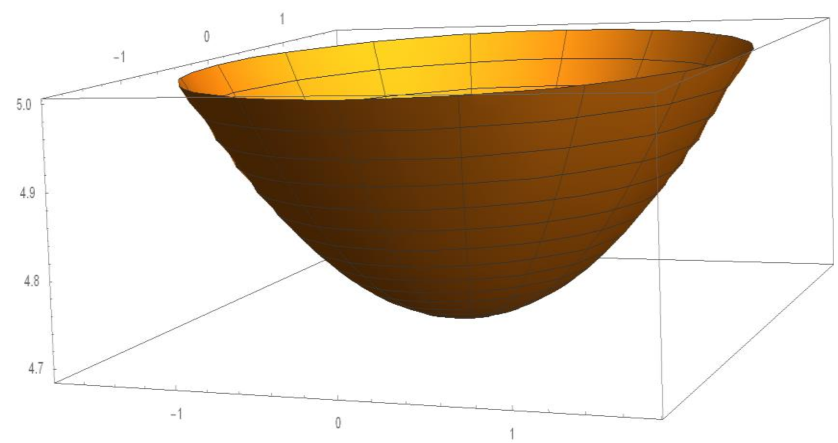
Find the volume above the upper cone

$z = \sqrt{x^2 + y^2}$ and

below $x^2 + y^2 + z^2 = 81$

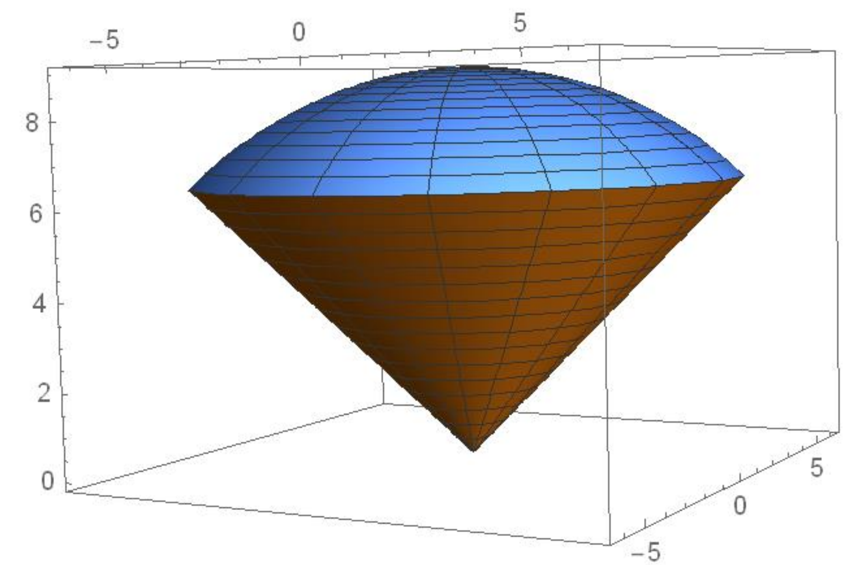
Volume enclosed by

$-x^2 - y^2 + z^2 = 22$ and $z = 5$.

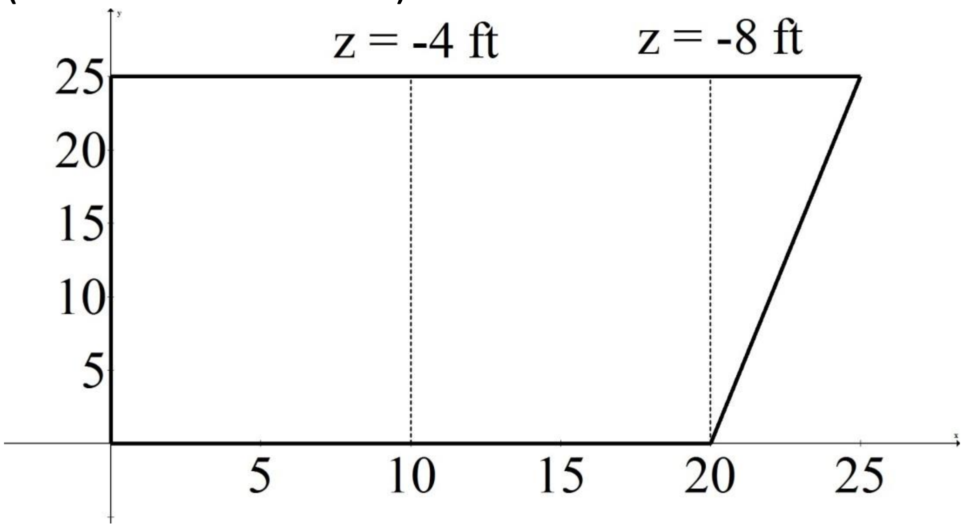


The volume above the upper cone $z = \sqrt{x^2 + y^2}$ and below

$x^2 + y^2 + z^2 = 81$



An applied problem:
Your swimming pool has the following shape (viewed from above)



The bottom of the pool is a plane with depths as indicated (the pool gets deeper in a linear way from left-to-right)

Solution:

1. Describe the surface (what is *z*?):
   Slope in y-direction = 0
   Slope in x-direction = -4/10 = -0.4
   Also the plane goes through (0, 0, 0)
   Thus, the plane that describes the bottom of the pool is: **z = -0.4x + 0y**
2. Describe the region in *xy*-plane:
   The line on the right goes through (20,0) and (25,25), so it has slope = 5 and it is given by the equation
   y = 5(x-20) = 5x – 100
   or **x = (y+100)/5 = 1/5 y + 20**
   The best way to describe this region is by thinking of it as a left-right region.
   On the left, we always have x = 0
   On the right, we always have x = 1/5 y + 20

Therefore, we have

$$\int_0^{25}\left(\int_0^{\frac{1}{5}y+20} -0.4\,x\,dx\right)dy = -741.\overline{6}\ \text{ft}^3$$